ECE 405/511
Error Control Coding

Decoding Binary BCH Codes

Decoding Binary BCH Codes

- $c(x)$ is the transmitted codeword
- $2t$ consecutive powers of α are roots

$$c(\alpha^{b}) = c(\alpha^{b+1}) = \cdots = c(\alpha^{b+2t-1}) = 0$$

- The received word is $r(x) = c(x)+e(x)$
- The error polynomial is

$$e(x) = e_0+e_1x+ \ldots +e_{n-1}x^{n-1}$$

- The syndromes are

$$S_j = r(\alpha^j) = e(\alpha^j) = \sum_{k=0}^{n-1} e_k (\alpha^j)^k, \qquad j = 1,\cdots,2t$$

Decoding Binary BCH Codes

- Suppose there are v errors in locations

$$i_1, i_2, \cdots, i_\nu$$

- The syndromes can be expressed in terms of these error locations

$$S_j = \sum_{l=1}^{v} e_{i_l} (\alpha^j)^{i_l} = \sum_{l=1}^{v} (\alpha^{i_l})^j = \sum_{l=1}^{v} X_l^j, \qquad j = 1, \cdots, 2t$$

- The X_l are the error locators
- The $2t$ syndrome equations can be expanded in terms of the v unknown error locations

Power-Sum Symmetric Equations

$$S_1 = X_1 + X_2 + \cdots + X_v$$

$$S_2 = X_1^2 + X_2^2 + \cdots + X_v^2$$

$$S_3 = X_1^3 + X_2^3 + \cdots + X_v^3$$

$$\vdots$$

$$S_{2t} = X_1^{2t} + X_2^{2t} + \cdots + X_v^{2t}$$

- The power-sum symmetric functions are nonlinear equations.
- Any method for solving these equations is a decoding algorithm for binary BCH codes.
- Peterson showed that these equations can be transformed into a series of linear equations.

The Error Locator Polynomial

- The error locator polynomial $\Lambda(x)$ has as its roots the inverses of the v error locators $\{X_l\}$

$$\Lambda(x) = \prod_{l=1}^{v}(1 - X_l x) = \Lambda_v x^v + \ldots + \Lambda_1 x + \Lambda_0$$

- The roots of $\Lambda(x)$ are then $X_1^{-1}, X_2^{-1}, \ldots, X_v^{-1}$
- Now express the coefficients of $\Lambda(x)$ in terms of the $\{X_l\}$ to get the elementary symmetric functions of the error locators

$$\Lambda_0 = 1$$

$$\Lambda_1 = \sum_{i=1}^{\nu} X_i = X_1 + X_2 + \cdots + X_{\nu-1} + X_\nu$$

$$\Lambda_2 = \sum_{i<j} X_i X_j = X_1 X_2 + X_1 X_3 + \cdots + X_{\nu-2} X_\nu + X_{\nu-1} X_\nu$$

$$\Lambda_3 = \sum_{i<j<k} X_i X_j X_k = X_1 X_2 X_3 + X_1 X_2 X_4 + \cdots + X_{\nu-2} X_{\nu-1} X_\nu$$

$$\vdots$$

$$\Lambda_\nu = \prod X_i = X_1 X_2 \cdots X_\nu$$

From these sets of equations we get Newton's Identities

$$S_1 + \Lambda_1 = 0$$

$$S_2 + \Lambda_1 S_1 + 2\Lambda_2 = 0$$

$$S_3 + \Lambda_1 S_2 + \Lambda_2 S_1 + 3\Lambda_3 = 0$$

$$\vdots$$

$$S_\nu + \Lambda_1 S_{\nu-1} + \cdots + \Lambda_{\nu-1} S_1 + \nu\Lambda_\nu = 0$$

$$S_{\nu+1} + \Lambda_1 S_\nu + \cdots + \Lambda_{\nu-1} S_2 + \Lambda_\nu S_1 = 0$$

$$\vdots$$

$$S_{2t} + \Lambda_1 S_{2t-1} + \cdots + \Lambda_{\nu-1} S_{2t-\nu+1} + \Lambda_\nu S_{2t-\nu} = 0$$

Binary BCH Codes

- In fields of characteristic 2, i.e. GF(2^m)

$$S_{2j} = \sum_{l=1}^{v} X_l^{2j} = \left(\sum_{l=1}^{v} X_l^{j} \right)^2 = S_j^2$$

thus every second equation in Newton's identities is redundant

Newton's Identities for Binary Codes

$$S_1 + \Lambda_1 = 0$$

$$S_3 + \Lambda_1 S_2 + \Lambda_2 S_1 + \Lambda_3 = 0$$

$$S_5 + \Lambda_1 S_4 + \Lambda_2 S_3 + \Lambda_3 S_2 + \Lambda_4 S_1 + \Lambda_5 = 0$$

$$\vdots$$

$$S_{2t-1} + \Lambda_1 S_{2t-2} + \Lambda_2 S_{2t-3} + \cdots + \Lambda_t S_{t-1} = 0$$

Peterson’s Direct Solution

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ S_2 & S_1 & 1 & 0 & \cdots & 0 & 0 \\ S_4 & S_3 & S_2 & S_1 & \cdots & 0 & 0 \\ S_6 & S_5 & S_4 & S_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ S_{2t-4} & S_{2t-5} & S_{2t-6} & S_{2t-7} & \cdots & S_{t-2} & S_{t-3} \\ S_{2t-2} & S_{2t-3} & S_{2t-4} & S_{2t-5} & \cdots & S_t & S_{t-1} \end{bmatrix} \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \\ \vdots \\ \Lambda_{t-1} \\ \Lambda_t \end{bmatrix} = \begin{bmatrix} -S_1 \\ -S_3 \\ -S_5 \\ -S_7 \\ \vdots \\ -S_{2t-3} \\ -S_{2t-1} \end{bmatrix}$$

$$\mathbf{A\Lambda} = \mathbf{S}$$

- If **A** is nonsingular, we can solve **AΛ** = **S** using linear algebra
- If there are *t*-1 or *t* errors, **A** has a nonzero determinant and a solution for **Λ** can be obtained
- If fewer than *t*-1 errors have occurred, delete the last two rows and the two rightmost columns of **A** and check again for singularity
- Continue until the remaining matrix is nonsingular

- There are two possibilities when a solution of $\mathbf{A\Lambda} = \mathbf{S}$ leads to an incorrect error locator polynomial
  1. If the received word is within Hamming distance t of an incorrect codeword, $\Lambda(x)$ will correct to that codeword, causing a decoding error
  2. If the received word is **not** within Hamming distance t of an incorrect codeword, $\Lambda(x)$ will not have the correct number of roots, or will have repeated roots, causing a decoding failure

Peterson's Algorithm

1. Compute the syndromes **S** from **r**.
2. Construct the syndrome matrix **A**.
3. Compute the determinant of **A**, if it is nonzero, go to 5.
4. Delete the last two rows and columns of **A** and go to 3.
5. Solve $\mathbf{A\Lambda} = \mathbf{S}$ to get $\Lambda(x)$.
6. Find the roots of $\Lambda(x)$, if there are an incorrect number of roots or repeated roots, declare a decoding failure.
7. Complement the bit positions in **r** indicated by $\Lambda(x)$. If fewer than t errors have been corrected, verify that the resulting codeword satisfies the syndrome equations. If not, declare a decoding failure.

Peterson’s Algorithm (Cont.)

- For simple cases, the equations can be solved directly
- Single error correction $\Lambda_1 = S_1$
- Double error correction

$$\Lambda_1 = S_1 \qquad \Lambda_2 = \frac{S_3 + S_1^3}{S_1}$$

- Triple error correction

$$\Lambda_1 = S_1 \qquad \Lambda_2 = \frac{S_1^2 S_3 + S_5}{S_1^3 + S_3} \qquad \Lambda_3 = \left(S_1^3 + S_3\right) + S_1\Lambda_2$$

Peterson's Algorithm (Cont.)

- Four error correction

$$\Lambda_1 = S_1 \qquad \Lambda_2 = \frac{S_1\left(S_7 + S_1^7\right) + S_3\left(S_1^5 + S_5\right)}{S_3\left(S_1^3 + S_3\right) + S_1\left(S_1^5 + S_5\right)}$$

$$\Lambda_3 = \left(S_1^3 + S_3\right) + S_1\Lambda_2 \qquad \Lambda_4 = \frac{\left(S_1^2 S_3 + S_5\right) + \left(S_1^3 + S_3\right)\Lambda_2}{S_1}$$

Example 9-1

- (31,21,5) 2 error correcting BCH code

  $g(x) = M_1(x)M_3(x) = (x^5+x^2+1)(x^5+x^4+x^3+x^2+1)$

  $= x^{10}+x^9+x^8+x^6+x^5+x^3+1$

  $\mathbf{r} = (0010000110011000000000000000000)$

  $r(\mathrm{x}) = x^2+x^7+x^8+x^{11}+x^{12}$

  $$S_1 = r(\alpha) = \alpha^7 \qquad S_2 = S_1^2 = \alpha^{14} \qquad S_3 = r(\alpha^3) = \alpha^8$$

  $$S_4 = S_1^4 = \alpha^{28}$$

Example 9-1 (Cont.)

- Double error correction

$$\Lambda_1 = S_1 = \alpha^7$$

$$\Lambda_2 = \frac{S_3 + S_1^3}{S_1} = \frac{\alpha^8 + \left(\alpha^7\right)^3}{\alpha^7} = \alpha^{15}$$

- Error locator polynomial

$\Lambda(x) = 1 + \alpha^7 x + \alpha^{15} x^2 \quad \rightarrow$ roots are α^{21} and α^{26}

- The error locators are $X_1 = \alpha^{-26} = \alpha^5$ and $X_2 = \alpha^{-21} = \alpha^{10}$

$$\Lambda(x) = (1 + \alpha^5 x)(1 + \alpha^{10} x)$$

Example 9-1 (Cont.)

r = (0010000011001100000000000000000)

e = (0000010000100000000000000000000)

c = (0010010110111000000000000000000)

check:

$c(x) = x^2 + x^5 + x^7 + x^8 + x^{10} + x^{11} + x^{12}$

$= x^2 g(x)$

so it is a valid codeword

Example 9-2

- In this example, the number of errors is less than the number of correctable errors

  $g(x) = 1+x+x^2+x^3+x^5+x^7+x^8+x^9+x^{10}+x^{11}+x^{15}$

  has 6 consecutive powers of α as roots

  $$\{\alpha,\alpha^2,\alpha^3,\alpha^4,\alpha^5,\alpha^6\}$$

  $r(x) = x^{10}$

$$S_1 = r(\alpha) = \alpha^{10} \quad S_2 = S_1^2 = \alpha^{20} \quad S_3 = r(\alpha^3) = \alpha^{30}$$

$$S_4 = S_1^4 = \alpha^{9} \quad S_5 = r(\alpha^5) = \alpha^{19} \quad S_6 = S_3^2 = \alpha^{29}$$

Example 9-2 (Cont.)

- The matrix **A** is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ \alpha^{20} & \alpha^{10} & 1 \\ \alpha^{9} & \alpha^{30} & \alpha^{20} \end{bmatrix}$$

- row 3 is equal to α^{20}×row 2
- Therefore remove the 2nd and 3rd rows and columns, giving

$$\mathbf{A} = \left[1\right]$$

- Thus $\Lambda_1 = S_1 = \alpha^{10}$ giving $X_1 = \alpha^{10}$ and $e(x) = x^{10}$
- $c(x) = r(x) + e(x) = x^{10} + x^{10} = 0$

Example 9-4 (Direct Solution)

- Triple error correcting BCH code n=31

  $g(x) = M_1(x)M_3(x)M_5(\mathrm{x}) =$
  $1+x+x^2+x^3+x^5+x^7+x^8+x^9+x^{10}+x^{11}+x^{15}$

  6 consecutive powers of α as roots $\{\alpha,\alpha^2,\alpha^3,\alpha^4,\alpha^5,\alpha^6\}$

  $r(x) = 1+x^9+x^{11}+x^{14}$

$$S_1 = r(\alpha) = 1 \quad S_2 = S_1^2 = 1 \quad S_3 = r(\alpha^3) = \alpha^{29}$$

$$S_4 = S_1^4 = 1 \quad S_5 = r(\alpha^5) = \alpha^{23} \quad S_6 = S_3^2 = \alpha^{27}$$

Example 9-4 (Cont.)

$$\Lambda_1 = S_1 = 1$$

$$\Lambda_2 = \frac{S_1^2 S_3 + S_5}{S_1^3 + S_3} = \alpha^{16} \qquad \Lambda_3 = \left(S_1^3 + S_3\right) + S_1 \Lambda_2 = \alpha^{17}$$

- Error locator polynomial

$$\Lambda(x) = 1 + x + \alpha^{16} x^2 + \alpha^{17} x^3$$

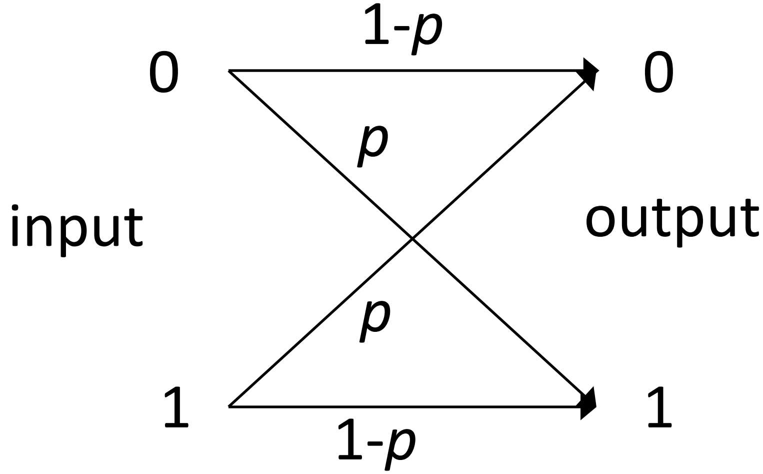
- The roots are α^{12}, α^{15}, α^{18}
- The errors are at locations 31-12=19, 31-15=16, 31-18=13
- $e(x) = x^{13}+x^{16}+x^{19}$
- $c(x) = r(x)+e(x) = 1+x^9+x^{11}+x^{13}+x^{14}+x^{16}+x^{19}$

Error Correction Procedure for BCH Codes

1. Compute the syndrome vector **S** = $(S_1, S_2, \ldots, S_{2t})$ from the received polynomial $r(x)$
2. Determine the error locator polynomial $\Lambda(x)$ from the syndromes $S_1, S_2, \ldots, S_{2t}$
3. Determine the error locators $X_1, X_2, \ldots, X_v$ by finding the roots of $\Lambda(x)$
4. Correct the errors in $r(x)$

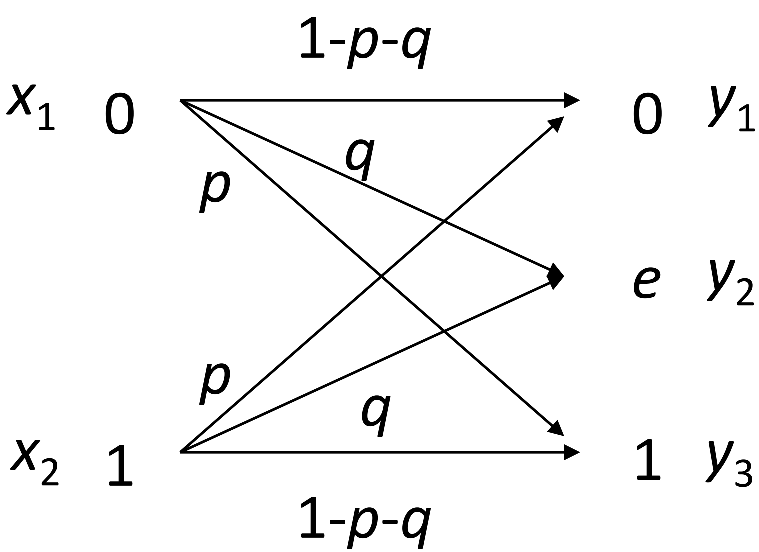
Binary Symmetric Channel

- Transmitted symbols are binary
- Errors affect 0s and 1s with equal probability (symmetric)
- Errors occur randomly and are independent from bit to bit (memoryless)



p is the probability of bit error – the crossover probability

Binary Errors with Erasure Channel



Error and Erasure Decoding

- e errors and f erasures can be corrected as long as

$$(2e+f) < d_{\text{min}}$$

- Thus we can correct twice as many erasures as we can errors
- This is because we know the locations of the erasures but not the locations of the errors

Error and Erasure Correction Procedure for BCH Codes

1. Put 0 in all erasure locations and decode.
2. Put 1 in all erasure locations and decode.
3. If only one decoding is successful, choose the resulting codeword.
4. If both are successful, choose the codeword with the smallest number of errors corrected outside the erased positions.